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MANIFOLDS OF NONANALYTICITY OF SOLUTIONS OF CERTAIN LINEAR PDE'S[1]

BY

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ABSTRACT. It is shown that if P is a linear partial differential operator with analytic coefficients, and if M is an analytic manifold of codimension 3 which is partially characteristic with respect to P and satisfies certain additional conditions, then one can find, in a neighborhood of any point of M, solutions of the equation $Pu = 0$ which are flat or singular precisely on M.

1. Introduction. Let $P = P(x, D)$ be a linear partial differential operator of order $m \geqslant 1$ with complex-valued coefficients which are defined and analytic in an open subset Ω of $\mathbf{R}^n$. Theorems 2 and 2′ of [1] assert that if M is an analytic manifold in Ω and if it is partially characteristic with respect to P and satisfies certain additional conditions, then one can find, in a neighborhood of any point of M, solutions of the homogeneous equation

$$P(x, D)u = 0 \tag{1.1}$$

which are (a) flat on M and analytic and nowhere vanishing in its complement, or (b) singular on M and analytic in its complement. In this paper we present new conditions on M and P under which these conclusions are still valid.

Let us recall some notation. By $p_m(x, \xi)$ we denote the principal symbol of P and set

$$A(x, \xi) = \operatorname{Re} p_m(x, \xi), \qquad B(x, \xi) = \operatorname{Im} p_m(x, \xi). \tag{1.2}$$

H_A and H_B are the Hamilton fields of A and B. For any point $\gamma = (x, \xi)$ in the cotangent space $T^*\Omega$, Hörmander's [2] integer $k = k(\gamma; A, B)$ is defined by

$$k = k(\gamma; A, B) = \sup\{j; j \in N; H_{C_1} \ldots H_{C_{l-1}} C_l(\gamma) = 0, \text{ for } 1 \leqslant l \leqslant j \text{ and } C_i = A \text{ or } B\}. \tag{1.3}$$

This integer is obviously invariant under symplectic transformations of coordinates and is also invariant under multiplication of p_m by a nonvanishing function. Note that $k \geqslant 1$ implies in particular that $p_m(x, \xi) = 0$. When $k = 3$ we also introduce the discriminant

$$D(\gamma; A, B) = \left[(H_A H_B H_A B)^2 - (H_A^2 H_A B)(H_B^2 H_A B)\right](\gamma) \tag{1.4}$$

the sign of which is also invariant under multiplication of p_m by a nonvanishing function (see Lemma 3.1).

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The result that we prove in this paper requires that the manifold M is of codimension 3. For simplicity in this introduction we state the result for the special case in which M is a point in $\mathbf{R}^3$.

"Let $x^0 \in \Omega \subset \mathbf{R}^3$ and suppose that for some $\xi^0 \in \mathbf{R}^3 \setminus 0$, the following conditions are satisfied:

(i) $k(x^0, \xi^0; A, B) = 3$;

(ii) $\operatorname{grad}_\xi A(x^0, \xi^0)$ and $\operatorname{grad}_\xi B(x^0, \xi^0)$ are linearly independent;

(iii) $D(x^0, \xi^0, A, B) < 0$.

Then in a neighborhood of x^0 one can find solutions of (1.1) which are flat or singular precisely at x^0."

In contrast, the corresponding result of **[1]** requires, in the present case in which (i) holds, that in place of (ii) and (iii) the following conditions are verified:

(ii)′ $\operatorname{grad}_\xi p_m(x^0, \xi^0) \neq 0$;

(iii)′ $H_A(x^0, \xi^0)$ and $H_B(x^0, \xi^0)$ are parallel.

Obviously if (iii)′ holds then (ii) cannot hold and it is easy to show that (iii)′ implies that $D(x^0, \xi^0; A, B) = 0$.

An illustration of the result of this paper is provided by the first order operator in $\mathbf{R}^3$,

$$L = \partial_t - i\partial_s - i\left(\tfrac{1}{3}t^3 + ts^2\right)\partial_x. \tag{1.5}$$

It is easy to check that conditions (i), (ii) and (iii) are verified if x^0 is given by $(x, t, s) = (0, 0, 0)$ and ξ^0 by $(\xi, \tau, \sigma) = (1, 0, 0)$. Obviously

$$\phi(x, t, s) = x - \tfrac{1}{3}s^3t + i\tfrac{1}{12}(t^4 + s^4) \tag{1.6}$$

satisfies the equation $L\phi = 0$; $\operatorname{Im} \phi(x, t, s) \geqslant 0$, and $\phi(x, t, s) = 0$ iff $(x, t, s) = (0, 0, 0)$. Consequently the functions

$$u_1(x, t, s) = \left[\phi(x, t, s)\right]^{1/2} \quad \text{and} \quad u_2(x, t, s) = \exp\left\{-1/\left[\phi(x, t, s)\right]^{1/5}\right\}$$

are solutions of $Lu = 0$, u_1 being analytic everywhere except at the origin and u_2 nonvanishing everywhere except at the origin where it is flat.

The precise statement of our theorem is given in §2. As in **[1]**, the proof is based on the construction of a complex phase function, i.e. a solution ϕ of the characteristic equation

$$p_m(x, \phi_x) = 0 \tag{1.7}$$

in some open neighborhood U of a point x^0 on the manifold M, possessing the following properties: (a) $\phi(x) = 0$ if and only if $x \in M \cap U$; (b) the values of ϕ in $\mathbf{C}$ avoid the negative imaginary half-axis; and (c) $\operatorname{grad} \phi \neq 0$. Once such a phase function has been constructed the end of the proof of the theorem proceeds exactly as in §6 of **[1]**.

The construction of the desired phase function consists of two parts. In the first part, the principal symbol p_m is reduced to a first order symbol. This is done in §3 where we also prove that the assumptions of our theorem are invariant under multiplication of p_m by a nonvanishing function. The phase function is actually constructed in §4 by solving a Cauchy problem for the reduced first order characteristic equation, with appropriately chosen Cauchy data containing a fourth order term with complex coefficient.

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2. Statement of results.

THEOREM 2.1. *Let M be an analytic manifold in Ω with* codim $M = 3$ *and suppose that there is an analytic hypersurface S containing M such that for every point $\gamma = (x, \xi) \in \Sigma$, where*

$$\Sigma = N^*(S)|_M = \{(x, \xi);\ x \in M, \xi \neq 0 \text{ normal to } S \text{ at } x\}, \tag{2.1}$$

the following conditions are satisfied:

$$\text{The dimension of the space spanned by } T_xM \text{ and by the base projections of } H_A(\gamma) \text{ and } H_B(\gamma) \text{ is } n-1. \tag{2.2}$$

$$k(\gamma; A, B) = 3. \tag{2.3}$$

$$D(\gamma; A, B) < 0. \tag{2.4}$$

Then, if x^0 is any point of M, the following conclusions hold:

(a) *There is an open neighborhood U of x^0 in Ω and a solution u of* (1.1) *in U such that $u \in C^\infty(U)$, u is flat on $M \cap U$ and u is analytic and nonvanishing in $U \setminus M$.*

(b) *For any integer $p \geqslant m$ there is an open neighborhood U_p of x^0 in Ω and a solution u of* (1.1) *in U_p such that $u \in C^p(U_p)$, u is analytic in $U_p \setminus M$ and u is not C^{p+1} in any neighborhood of any point of $M \cap U_p$.*

REMARK. It is easy to show that assumption (2.4) implies that $H_A(\gamma)$ and $H_B(\gamma)$ are not parallel. This however does not necessarily imply assumption (2.2) (even when M is a point).

3. Invariance of the assumptions and reduction of the principal symbol.

LEMMA 3.1. *Let A, B, f_1, f_2, f_3, f_4 be real-valued C^∞ functions defined near the point $\gamma \in T^*\Omega$. Suppose that $f_1f_4 - f_2f_3 \neq 0$ and set*

$$\tilde{A} = f_1A + f_2B, \qquad \tilde{B} = f_3A + f_4B.$$

Then

$$k(\gamma; \tilde{A}, \tilde{B}) = k(\gamma; A, B) \tag{3.1}$$

and, if $k(\gamma; A, B) \geqslant 3$,

$$D(\gamma; \tilde{A}, \tilde{B}) = (f_1f_4 - f_2f_3)^4 D(\gamma; A, B). \tag{3.2}$$

PROOF. (3.1) was proved in [**1**]. (3.2) is obtained by a straightforward calculation the length of which is considerably reduced by remembering that $k \geqslant 3$ means that at γ

$$A = B = H_AB = H_A^2B = H_BH_AB = 0,$$

and by ignoring terms which give zero contribution at the "3-H" level.

An immediate consequence of Lemma 3.1 is that assumptions (2.3) and (2.4) are invariant under multiplication of p_m by a nonvanishing complex-valued function.

Next, we turn to the proof of invariance of assumption (2.2) (assuming that $p_m|_\Sigma = 0$). We find it convenient to combine this with the beginning of the reduction of the principal symbol.

LEMMA 3.2. *Suppose that the assumptions of Theorem* 2.1 *hold except that in place of* (2.3) *and* (2.4) *we only assume that*

$$p_m|_\Sigma = 0. \tag{3.3}$$

Then (i) *assumption* (2.2) *is invariant under multiplication of* p_m *by a nonvanishing function, and*

(ii) *in a neighborhood* $U \subset \Omega$ *of any point of* M *we can make an analytic change of variables with the new variables centered at that point and denoted by*

$$(x, t, s, y) = (x, t, s, y_1, \ldots, y_l), \qquad 3 + l = n,$$

such that in U, S *is given by* $x = 0$ *and* M *by*

$$M: x = t = s = 0 \tag{3.4}$$

and in T^*U, Σ *is given by*

$$\Sigma: x = t = s = 0, \qquad \tau = \sigma = 0, \qquad \eta = 0. \tag{3.5}$$

Moreover, in a neighborhood of the point $\gamma^0 \in \Sigma$ *given by*

$$\gamma^0: x = t = s = 0, \quad y = 0, \quad \xi = 1, \quad \tau = \sigma = 0, \quad \eta = 0 \tag{3.6}$$

the principal symbol p_m *can be written in the form*

$$\begin{aligned} p_m(x, t, s, y, \xi, \tau, \sigma, \eta) = {} & q(x, t, s, y, \xi, \tau, \sigma, \eta) \\ & \cdot \{\tau - a(x, t, s, y, \xi, \sigma, \eta) - ir(x, s, y, \xi, \sigma, \eta) \\ & \cdot [\sigma - e(x, s, y, \xi, \sigma, \eta)] + ig(x, t, s, y, \xi, \sigma, \eta)\} \end{aligned} \tag{3.7}$$

where q, a, r, e *and* g *are analytic functions of their arguments, homogeneous with respect to* ξ, τ, σ, η *of degree* $m - 1$ *for* q, 1 *for* a, e, g *and* 0 *for* r; q *is complex valued and nonvanishing while* a, r, e *and* g *are real valued and satisfy*

$$a(x, t, s, y, 1, 0, 0) = 0, \qquad \operatorname{grad} a(x, t, s, y, 1, 0, 0) = 0, \tag{3.8}$$

$$e(x, s, y, 1, 0, 0) = 0, \qquad \operatorname{grad} e(x, s, y, 1, 0, 0) = 0, \tag{3.9}$$

$$r(x, s, y, 1, 0, 0) = 1, \tag{3.10}$$

$$g(x, t, s, y, \xi, \sigma, \eta)|_{t=0} = 0. \tag{3.11}$$

PROOF. Clearly we can introduce new local coordinates (x, t, s, y) centered at any given point of M so that S is given by $x = 0$ and M and Σ by (3.4) and (3.5) respectively. Since ξ is free on Σ, we have from (3.3),

$$\partial_\xi p_m|_\Sigma = 0. \tag{3.12}$$

In view of (3.12) assumption (2.2) means that

$$(A_\tau \partial_t + A_\sigma \partial_s)|_\Sigma \quad \text{and} \quad (B_\tau \partial_t + B_\sigma \partial_s)|_\Sigma$$

are linearly independent and hence

$$(A_\tau B_\sigma - A_\sigma B_\tau)|_\Sigma \neq 0. \tag{3.13}$$

Now let $q = \alpha + i\beta$ be a nonvanishing function of all variables in the region under consideration (α, β real) and set

$$\tilde{p}_m = qp_m = (\alpha A - \beta B) + i(\beta A + \alpha B) = \tilde{A} + i\tilde{B}. \tag{3.14}$$

Using (3.3), (3.13) and the nonvanishing of q we easily find that

$$(\tilde{A}_\tau \tilde{B}_\sigma - \tilde{A}_\sigma \tilde{B}_\tau)|_\Sigma = \left[(\alpha^2 + \beta^2)(A_\tau B_\sigma - A_\sigma B_\tau)\right]|_\Sigma \neq 0. \tag{3.15}$$

But (3.15) implies that

$$(\tilde{A}_\tau \partial_t + \tilde{A}_\sigma \partial_s)|_\Sigma \quad \text{and} \quad (\tilde{B}_\tau \partial_t + \tilde{B}_\sigma \partial_s)|_\Sigma$$

are linearly independent, which in turn means that (2.2) holds also for $\tilde{p}_m$. Assertion (i) of Lemma 3.2 is proved.

Let us turn now to the proof of assertion (ii). From (3.13) we have

$$\partial_\tau p_m|_\Sigma \neq 0, \qquad \partial_\sigma p_m|_\Sigma \neq 0. \tag{3.16}$$

In view of (3.3) and (3.16) we can apply the implicit function theorem to obtain the following factorization of p_m near γ^0,

$$p_m(x, t, s, y, \xi, \tau, \sigma, \eta) = q(x, t, s, y, \xi, \tau, \sigma, \eta)\left[\tau - \lambda(x, t, s, y, \xi, \sigma, \eta)\right] \tag{3.17}$$

where q and λ are analytic functions of their arguments, homogeneous with respect to ξ, τ, σ, η of degree $m - 1$ and 1 respectively, and q is nonvanishing. Moreover, if

$$a = \operatorname{Re} \lambda \quad \text{and} \quad b = \operatorname{Im} \lambda \tag{3.18}$$

we must also have from (3.3)

$$a|_\Sigma = 0, \qquad b|_\Sigma = 0. \tag{3.19}$$

Since $\tau - a - ib = q^{-1}p_m$ with $q^{-1} \neq 0$, we must also have from (3.15)

$$\partial_\sigma b|_\Sigma \neq 0. \tag{3.20}$$

Hence we can again apply the implicit function theorem to obtain the following factorization of $b|_{t=0}$ near γ^0:

$$b(x, 0, s, y, \xi, \sigma, \eta) = r(x, s, y, \xi, \sigma, \eta)\left[\sigma - c(x, s, y, \xi, \eta)\right] \tag{3.21}$$

where r and c are real-valued analytic functions of their arguments, homogeneous with respect to ξ, σ, η of degree 0 and 1 respectively and r is nonvanishing.

We now use Lemma 4.1 of [1] to make a local analytic change of the x, s, y variables, pointwise preserving $s = 0$, so that in terms of the new (symplectic) coordinates which we will denote by the same letters, the function $\sigma - c$ still has the same form, but now c satisfies

$$c(x, s, y, 1, 0) = 0, \qquad \operatorname{grad} c(x, s, y, 1, 0) = 0. \tag{3.22}$$

It is easy to check that in the new coordinates, M and Σ are still given by (3.4) and (3.5).

Next we make another analytic change of the x, s, y variables in order to simplify the function r. Let the new variables $\tilde{x}, \tilde{s}, \tilde{y}$ be given by

$$\tilde{x} = x, \qquad \tilde{s} = \phi(x, s, y), \qquad \tilde{y} = y \tag{3.23}$$

where ϕ is the solution of the Cauchy problem

$$r(x, s, y, 1, 0, 0)\frac{\partial \phi}{\partial s} = 1, \qquad \phi(x, 0, y) = 0. \tag{3.24}$$

Since

$$\xi = \tilde{\xi} + \frac{\partial \phi}{\partial x}\tilde{\sigma}, \qquad \sigma = \frac{\partial \phi}{\partial s}\tilde{\sigma}, \qquad \eta = \tilde{\eta} + \frac{\partial \phi}{\partial y}\tilde{\sigma},$$

the function $b|_{t=0} = r(\sigma - c)$ becomes, in the new symplectic coordinates,

$$r\left(\tilde{x}, s(\tilde{x}, \tilde{s}, \tilde{y}), \tilde{y}, \tilde{\xi} + \frac{\partial \phi}{\partial x}\tilde{\sigma}, \frac{\partial \phi}{\partial s}\tilde{\sigma}, \tilde{\eta} + \frac{\partial \phi}{\partial y}\tilde{\sigma}\right)$$
$$\cdot\left[\frac{\partial \phi}{\partial s}\tilde{\sigma} - c\left(\tilde{x}, s(\tilde{x}, \tilde{s}, \tilde{y}), \tilde{y}, \tilde{\xi} + \frac{\partial \phi}{\partial x}\tilde{\sigma}, \tilde{\eta} + \frac{\partial \phi}{\partial y}\tilde{\sigma}\right)\right]$$
$$= \tilde{r}(\tilde{x}, \tilde{s}, \tilde{y}, \tilde{\xi}, \tilde{\sigma}, \tilde{\eta})\left[\tilde{\sigma} - \tilde{e}(\tilde{x}, \tilde{s}, \tilde{y}, \tilde{\xi}, \tilde{\sigma}, \tilde{\eta})\right]$$

with the obvious definitions of the functions $\tilde{r}$ and $\tilde{e}$. It follows immediately from (3.24) that the function $\tilde{r}$ satisfies (3.10) after dropping the tildes. Similarly, it follows from (3.22) that the function $\tilde{e}$ satisfies (3.9) again after dropping the tildes. Again it is easy to check that in the new coordinates, M and Σ are still given by (3.4) and (3.5).

Let us recapitulate briefly. In a neighborhood of γ^0 we have written p_m in the form

$$p_m = q(\tau - a - ib) \quad \text{with } b|_{t=0} = r(\sigma - e),$$

where the functions e and r satisfy (3.9) and (3.10) respectively. In order to complete the proof of assertion (ii) of the lemma we again apply Lemma 4.1 of **[1]** to make a local analytic change of all variables, pointwise preserving $t = 0$, so that in the new (symplectic) coordinates, the function $\tau - a$ still has the same form, but now the function a satisfies (3.8). Again it is easy to check that in the new coordinates, M and Σ are still given by (3.4), (3.5). Moreover, since the new and old variables, x, s, y, ξ, σ, η, are equal when $t = 0$, the function $b|_{t=0}$ is still the same function in terms of the new coordintes, and hence the previously achieved simplification for it is not disturbed. The proof of Lemma 3.2 is now complete.

LEMMA 3.3. *Suppose that the assumptions of Theorem* 2.1 *hold and let*

$$G(t, s, y) = g_t(0, t, s, y, 1, 0, 0) \tag{3.25}$$

where g is the function appearing in the representation (3.7) *of* p_m. *Then G satisfies*

$$G(0, 0, y) = G_t(0, 0, y) = G_s(0, 0, y) = 0 \tag{3.26}$$

and

$$\left[G_{ts}(0, 0, y)\right]^2 - \left[G_{tt}(0, 0, y)\right]\left[G_{ss}(0, 0, y)\right] < 0. \tag{3.27}$$

PROOF. Since the assumptions of the theorem are invariant under multiplication of p_m by a nonvanishing function, we may ignore the factor q in the representation (3.7) of p_m. We have

$$A = \tau - a(x, t, s, y, \xi, \sigma, \eta),$$
$$B = -r(x, s, y, \xi, \sigma, \eta)\left[\sigma - e(x, s, y, \xi, \sigma, \eta)\right] + g(x, t, s, y, \xi, \sigma, \eta).$$

Hence

$$H_A = \partial_t - H_a, \qquad H_B = -r\partial_s - \sigma H_r + H_{re} + H_g$$

and $H_A B = g_t - H_a B$. Now, from (3.8), (3.9) and (3.10) we have

$$H_{\partial_s^j \partial_t^l a}|_\Sigma = H_{\partial_s^j (re)}|_\Sigma = 0, \qquad j, l = 0, 1, 2, \ldots, \tag{3.28}$$

and

$$r|_\Sigma = 1, \qquad \partial_s^j r|_\Sigma = 0, \qquad j = 1, 2, \ldots. \tag{3.29}$$

It follows easily from (3.28), (3.29) and (3.11) that

$$H_A^j|_\Sigma = \partial_t^j, \qquad j = 1, 2, \ldots, \tag{3.30}$$

and

$$H_B^j|_\Sigma = (-1)^j \partial_s^j, \qquad j = 1, 2, \ldots, \tag{3.31}$$

with (3.31) valid only if acting on functions independent of τ. Hence

$$H_A^j(H_A B)|_\Sigma = \partial_t^j g_t(0, 0, 0, y, 1, 0, 0)\xi, \qquad j = 0, 1, \ldots, \tag{3.32}$$

$$H_B^j(H_A B)|_\Sigma = (-1)^j \partial_s^j g_t(0, 0, 0, y, 1, 0, 0)\xi, \qquad j = 0, 1, \ldots. \tag{3.33}$$

Now

$$H_A H_A B = (g_t)_t - H_{a_t} B - H_a B_t - H_a(H_A B)$$

and using (3.28) and (3.31)

$$H_B H_A(H_A B)|_\Sigma = -(g_t)_{ts}(0, 0, 0, y, 1, 0, 0)\xi. \tag{3.34}$$

Assertions (3.26) and (3.27) of the lemma follow easily using (3.32), (3.33), (3.34), the identity $H_A H_B H_A B = H_B H_A H_A B$, and assumptions (2.3) and (2.4).

REMARK. Formulas (3.32), (3.33) and (3.34) were obtained using only the assumptions of Lemma 3.2.

We are now ready to construct the desired phase function by solving the characteristic equation (1.7). We can of course ignore the factor q in the representation (3.7) of p_m.

4. Construction of the phase function.

LEMMA 4.1. *Suppose that the functions a, r, e and g have the properties described in Lemmas 3.2 and 3.3. Then there are real-valued analytic functions $C_1 = C_1(y)$ and $C_2 = C_2(y)$ defined near $y = 0$, such that the solution of the Cauchy problem*

$$\phi_t - a(x, t, s, y, \phi_x, \phi_s, \phi_y) - ir(x, s, y, \phi_x, \phi_s, \phi_y)\phi_s$$
$$+ i(re)(x, s, y, \phi_x, \phi_s, \phi_y) + ig(x, t, s, y, \phi_x, \phi_s, \phi_y) = 0, \tag{4.1}$$

$$\phi(x, 0, s, y) = x + ix^2 + [C_1(y) + iC_2(y)]s^4 \tag{4.2}$$

has the property

$$\operatorname{Re} \phi(x, t, s, y) = 0 \textit{ implies } \operatorname{Im} \phi(x, t, s, y) \geqslant K(x^2 + t^4 + s^4) \tag{4.3}$$

in some neighborhood of the origin of $\mathbf{R}^n$, for some constant $K > 0$.

PROOF. From Lemma 3.3 we know that

$$g(0, t, s, y, 1, 0, 0) = g_3(t, s; y) + (\text{terms of degree} \geqslant 4 \text{ in } t, s) \tag{4.4}$$

where g_3 is a homogeneous polynomial of degree 3 in t, s with analytic coefficients in y,

$$g_3(0, s; y) = 0 \tag{4.5}$$

and $\partial_t g_3(t, s; y)$ is a definite quadratic form in t, s. By replacing $\xi = 1$ by $\xi = -1$ in the definition (3.6) of the point γ^0, we may assume without loss of generality that $-\partial_t g_3(t, s, y)$ is positive definite. We know then that there are analytic functions $\alpha = \alpha(y)$, $\beta = \beta(y)$ with

$$\alpha^2 + \beta^2 = 1 \tag{4.6}$$

and positive analytic functions $Q = Q(y)$ and $R = R(y)$ such that

$$-\partial_t g_3(t, s; y) = Q(\alpha t - \beta s)^2 + R(\beta t + \alpha s)^2. \tag{4.7}$$

Integrating (4.7) and using (4.5) we find that

$$-g_3(t, s; y) = \frac{Q}{3\alpha}(\alpha t - \beta s)^3 + \frac{R}{3\beta}(\beta t + \alpha s)^3 + \frac{Q\beta^3}{3\alpha}s^3 - \frac{R\alpha^3}{3\beta}s^3. \tag{4.8}$$

Expanding the solution ϕ of (4.1), (4.2) in powers of x, we have

$$\phi(x, t, s, y) = x + ix^2 + \phi_0(t, s, y) + \phi_1(t, s, y)x + O(x^2|t|) \tag{4.9}$$

where

$$\phi_0(0, s, y) = (C_1 + iC_2)s^4, \qquad \phi_1(0, s, y) = 0. \tag{4.10}$$

Substituting (4.9) into (4.1) and setting $x = 0$ we obtain

$$\begin{aligned} &\frac{\partial\phi_0}{\partial t} - a\left(0, t, s, y, 1 + \phi_1, \frac{\partial\phi_0}{\partial s}, \frac{\partial\phi_0}{\partial y}\right) \\ &\quad - ir\left(0, s, y, 1 + \phi_1, \frac{\partial\phi_0}{\partial s}, \frac{\partial\phi_0}{\partial y}\right)\frac{\partial\phi_0}{\partial s} \\ &\quad + i(re)\left(0, s, y, 1 + \phi_1, \frac{\partial\phi_0}{\partial s}, \frac{\partial\phi_0}{\partial y}\right) \\ &\quad + ig\left(0, t, s, y, 1 + \phi_1, \frac{\partial\phi_0}{\partial s}, \frac{\partial\phi_0}{\partial y}\right) = 0. \end{aligned} \tag{4.11}$$

Expanding the functions a, r, re and g about $\sigma = 0$, $\eta = 0$, and using (3.8)–(3.11) and homogeneity, we can write equation (4.11) in the form

$$\begin{aligned} &\partial\phi_0/\partial t - i\partial\phi_0/\partial s + ig(0, t, s, y, 1, 0, 0)(1 + \phi_1) \\ &\quad + (\text{terms with second degree factors in } \partial\phi_0/\partial s \text{ and } \partial\phi_0/\partial y) \\ &\quad + (\text{terms with factors } t\partial\phi_0/\partial s \text{ or } t\partial\phi_0/\partial y) = 0. \end{aligned} \tag{4.12}$$

We can now easily conclude from (4.12) that ϕ_0 begins with fourth order terms in t and s. Indeed from (4.10) we have

$$\begin{aligned} \phi_0(t, s, y) &= a_{10}t + a_{20}t^2 + a_{11}ts + a_{30}t^3 + a_{21}t^2 s \\ &\quad + a_{12}ts^2 + (\text{terms of degree} \geqslant 4), \end{aligned}$$

and substituting into (4.12) and equating to zero the coefficients of degree ≤ 2 in t and s we find that $a_{10} = a_{20} = a_{11} = a_{30} = a_{21} = a_{12} = 0$. Hence

$$\phi_0(t, s, y) = u(t, s, y) + iv(t, s, y) + (\text{terms of degree} \geq 5 \text{ in } t \text{ and } s) \tag{4.13}$$

where u and v are real and homogeneous of degree 4 in t and s with coefficients functions of y. Substituting (4.13) into (4.12) and retaining only the terms of degree 3 in t and s we obtain

$$\frac{\partial}{\partial t}(u + iv) - i\frac{\partial}{\partial s}(u + iv) + ig_3 = 0. \tag{4.14}$$

Also from (4.10) we have

$$(u + iv)|_{t=0} = (C_1 + iC_2)s^4. \tag{4.15}$$

Equations (4.14) and (4.15) uniquely determine $u + iv$ in terms of C_1 and C_2 which are still to be assigned. However, instead of assigning values to C_1 and C_2 we will first choose v so that it has the desired properties and then determine u, C_1 and C_2 so that (4.14) and (4.15) are satisfied.

Separating the real and imaginary parts in (4.14) we get

$$u_t + v_s = 0, \tag{4.16}$$

$$v_t - u_s = -g_3, \tag{4.17}$$

and eliminating u between these two equations, we find that v must satisfy

$$v_{tt} + v_{ss} = -\frac{\partial}{\partial t} g_3. \tag{4.18}$$

Using (4.7), equation (4.18) becomes

$$v_{tt} + v_{ss} = Q(\alpha t - \beta s)^2 + R(\beta t + \alpha s)^2. \tag{4.19}$$

From the invariance of the Laplacian under rotation (or the fact that $\alpha^2 + \beta^2 = 1$) we find that

$$v(t, s, y) = \frac{Q}{12}(\alpha t - \beta s)^4 + \frac{R}{12}(\beta t + \alpha s)^4 \tag{4.20}$$

satisfies (4.19). Since this v has the desired properties, we choose it and then determine u, C_1 and C_2 so that (4.15), (4.16) and (4.17) are satisfied. From (4.15) and (4.20) we must have

$$C_2 = \frac{Q}{12}\beta^4 + \frac{R}{12}\alpha^4. \tag{4.21}$$

From (4.16) and (4.20) we have

$$u_t = \frac{Q\beta}{3}(\alpha t - \beta s)^3 - \frac{R\alpha}{3}(\beta t + \alpha s)^3. \tag{4.22}$$

Integrating (4.22) and using (4.15) we obtain

$$u(t, s, y) = C_1 s^4 + \frac{Q}{12}\frac{\beta}{\alpha}(\alpha t - \beta s)^4 - \frac{R}{12}\frac{\alpha}{\beta}(\beta t + \alpha s)^4 - \frac{Q}{12}\frac{\beta}{\alpha}\beta^4 s^4 + \frac{R}{12}\frac{\alpha}{\beta}\alpha^4 s^4. \tag{4.23}$$

Finally substituting (4.8), (4.20) and (4.23) into (4.17) we find that the resulting equation will be satisfied if

$$C_1 = -\frac{Q}{12}\alpha\beta^3 + \frac{R}{12}\beta\alpha^3. \tag{4.24}$$

Indeed as a check we verify that (4.20), (4.21), (4.23) and (4.24) satisfy (4.15)–(4.17).

We can now easily complete the proof of the lemma. From (4.9) we have

$$\operatorname{Re}\phi = x(1 + \operatorname{Re}\phi_1) + \operatorname{Re}\phi_0 + O(x^2|t|), \tag{4.25}$$

$$\operatorname{Im}\phi = x^2 + \operatorname{Im}\phi_0 + (\operatorname{Im}\phi_1)x + O(x^2|t|). \tag{4.26}$$

From (4.13), (4.20) and (4.23) we have

$$\operatorname{Re}\phi_0 = (\text{terms of degree} \geqslant 4 \text{ in } t \text{ and } s), \tag{4.27}$$

$$\operatorname{Im}\phi_0 \geqslant K_1(t^4 + s^4) \tag{4.28}$$

where K_1 is a positive constant. From (4.25) and (4.27) it follows that when $\operatorname{Re}\phi = 0$ then

$$x = (\text{terms of degree} \geqslant 4 \text{ in } t \text{ and } s) + O(x^2|t|). \tag{4.29}$$

Now substituting (4.29) into (4.26) and using (4.10) and (4.28), assertion (4.3) of Lemma 4.1 follows immediately.

References

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